HORNSBY GIRLS HIGH SCHOOL

Mathematics Extension 1

Year 12 Higher School Certificate

Trial Examination Term 3 2020

STUDENT NUMBER: ____________________

STUDENT NAME: ____________________ **TEACHER NAME:** ____________________

General Instructions:

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/ or calculations

Total Marks: 70

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7–12)

- Attempt Questions 11–14
- Start each question in a new writing booklet
- Write your student number on every writing booklet
- Allow about 1 hour and 45 minutes for this section

Question	*1-10*	*11*	*12*	*13*	*14*	*Total*
Total	*/10*	*/15*	*/15*	*/15*	*/15*	*/70*

This assessment task constitutes 30% of the Higher School Certificate Course School Assessment

BLANK PAGE

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1–10.

---

1. Let $P(x) = x^2 + bx + c$ where b and c are constants. The zeros of $P(x)$ are α and $\alpha + 1$.

What are the correct expressions for b and c in terms of α ?

(A) $b = -(2\alpha + 1)$ and $c = \alpha^2 + \alpha$

(B) $b = 2\alpha + 1$ and $c = \alpha^2 + \alpha$

(C) $b = \alpha^2 + \alpha$ and $c = -(2\alpha + 1)$

(D) $b = \alpha^2 + \alpha$ and $c = 2\alpha + 1$

2. What is the derivative of $\tan^{-1}(2x-1)$?

(A) $\dfrac{1}{4x^2 - 4x + 2}$

(B) $\dfrac{2x-1}{2x^2 - 2x + 1}$

(C) $\dfrac{2}{2x^2 - 2x + 1}$

(D) $\dfrac{1}{2x^2 - 2x + 1}$

3. 80 students are allocated at random into different study groups. Determine the number of groups required so that **at least one group** has **9 or more** students.

(A) 11

(B) 10

(C) 9

(D) 8

4. Which of the following expressions is equal to $\cos x + \sin x$?

(A) $\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$

(B) $2\sin\left(x+\frac{\pi}{4}\right)$

(C) $\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$

(D) $2\sin\left(x-\frac{\pi}{4}\right)$

5. Four males and four females are to sit around a table. In how many ways can this be done if the males and females alternate?

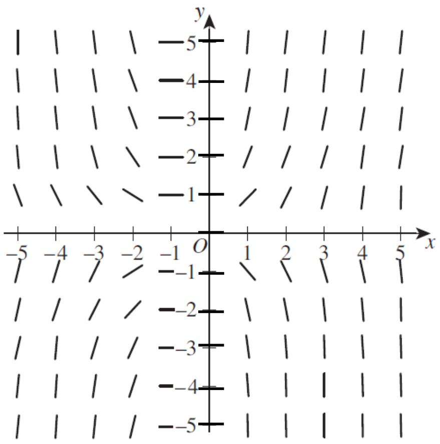
(A) 144

(B) 2880

(C) 5040

(D) 40 320

6. The direction (slope) field for a first order differential equation is shown.



Which of the following could be the differential equation represented?

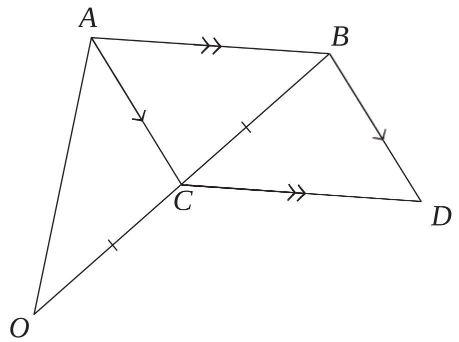
(A) $\frac{dy}{dx}=(x+1)^3$

(B) $\frac{dy}{dx}=x(y+1)$

(C) $\frac{dy}{dx}=(x+1)y$

(D) $\frac{dy}{dx}=(x-1)y$

7. The position vectors of points A and B are $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively. Point C is the midpoint of OB and point D is such that $ABDC$ is a parallelogram.



Which of the following is the position vector of D?

(A) $\frac{3}{2}\underset{\sim}{b}+\underset{\sim}{a}$

(B) $\frac{3}{2}\underset{\sim}{b}-\underset{\sim}{a}$

(C) $\frac{1}{2}\underset{\sim}{b}-\frac{1}{2}\underset{\sim}{a}$

(D) $\frac{1}{2}\underset{\sim}{b}-\underset{\sim}{a}$

8. What is the value of k such that the function $f(x)=\frac{k}{1+x^2}$, $-1\le x\le 1$ is a probability density function?

(A) $k=\frac{\pi}{4}$

(B) $k=\frac{\pi}{2}$

(C) $k=\frac{2}{\pi}$

(D) $k=\frac{4}{\pi}$

9. A curve C has parametric equations $x=\cos^2 t$ and $y=4\sin^2 t$ for $t\in\mathbb{R}$.
What is the Cartesian equation of C?

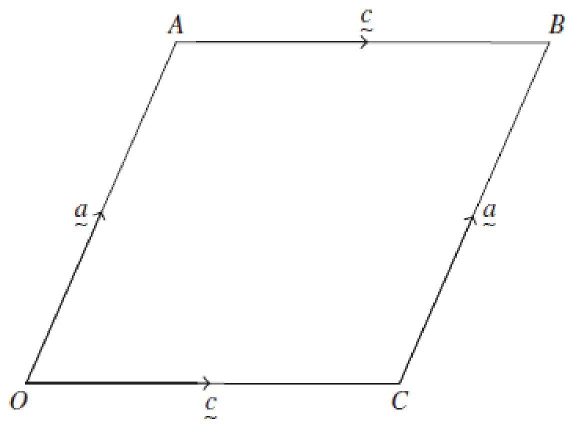
(A) $y=1-x$ for $0\le x\le 1$

(B) $y=4-4x$ for $t\in\mathbb{R}$

(C) $y=4-4x$ for $0\le x\le 1$

(D) $y=1-x$ for $t\in\mathbb{R}$

10. The diagram shows *OABC*, a rhombus in which $\overrightarrow{OA} = \overrightarrow{CB} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \overrightarrow{AB} = \underset{\sim}{c}$.



To prove that the diagonals of OABC are perpendicular, it is required to show that

(A) $(\underset{\sim}{a}+\underset{\sim}{c})\cdot(\underset{\sim}{a}+\underset{\sim}{c})=0$

(B) $(\underset{\sim}{a}-\underset{\sim}{c})\cdot(\underset{\sim}{a}-\underset{\sim}{c})=0$

(C) $(\underset{\sim}{a}-\underset{\sim}{c})\cdot(\underset{\sim}{a}+\underset{\sim}{c})=0$

(D) $\underset{\sim}{a}\cdot\underset{\sim}{c}=0$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a new writing booklet. Extra writing booklets are available.
In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new writing booklet

(a) Consider the function $f(x) = x^2 - 4x + 6$.

(i) Explain why the domain of $f(x)$ must be restricted if $f(x)$ is to have an inverse function. **1**

(ii) Given that the domain of $f(x)$ is restricted to $x \le 2$, find an expression for $f^{-1}(x)$. **2**

(iii) Given the restriction in part (a) (ii), state the domain and range of $f^{-1}(x)$. **2**

(iv) The curve $y = f(x)$ $(x \le 2)$ and the curve $y = f^{-1}(x)$ intersect at the point P. **2**

Find the coordinates of P.

(b) Use the substitution $u = 1 + 2\tan x$ to evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{(1+2\tan x)^2 \cos^2 x}\, dx$. **2**

(c) Use t-formulae to solve the equation $\cos x - \sin x = -1$, where $0 \le x \le 2\pi$. **3**

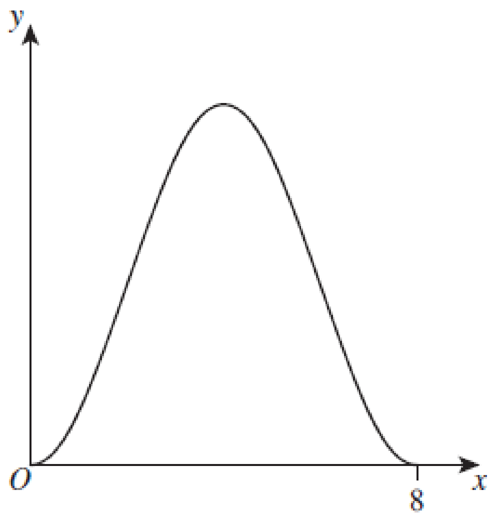
(d) Use the substitution $u = 1 - x$ to evaluate $\int_{-3}^{0} \frac{x}{\sqrt{1-x}}\, dx$ **3**

End of Question 11

Question 12 (15 marks) Start a new writing booklet

(a) Evaluate $\int_0^{\frac{2}{3}} \frac{1}{\sqrt{4-9x^2}}\, dx$. **3**

(b) A proposed plan for a garden is shown in the diagram. The curved boundary of the garden is modelled by the function $f(x) = 6\sin^2\left(\frac{\pi x}{8}\right)$, $0 \le x \le 8$.



(i) Use the identity $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$ to show that $\sin^2\left(\frac{\pi x}{8}\right) = \frac{1}{2}\left(1 - \cos\frac{\pi x}{4}\right)$. **2**

(ii) Use the result from part (a) (i) to find the area A of the garden. **2**

Question 12 continues on page 9

Question 12

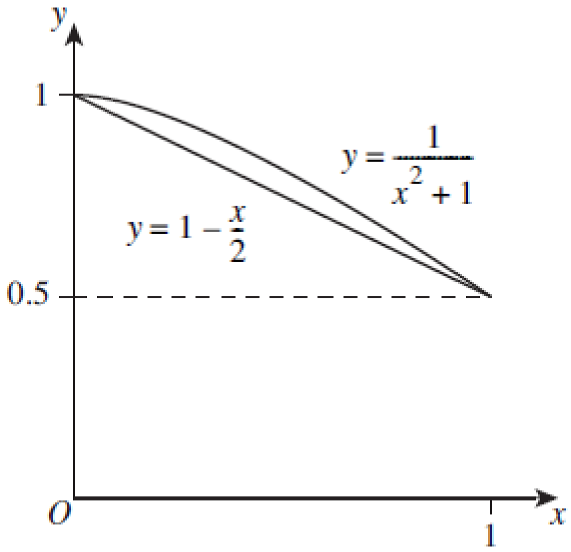
(c) The work done W, by a constant force $\underset{\sim}{F}$, in moving a particle through a displacement $\underset{\sim}{s}$, is defined by the formula $W = \underset{\sim}{F} \cdot \underset{\sim}{s}$. A force described by the vector $\underset{\sim}{F} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ moves a particle along the line l from $P\ (-1, 2)$ to $Q\ (2, -2)$ **so that** $\underset{\sim}{s} = \overrightarrow{PQ}$.

(i) Find $\underset{\sim}{s}$ and hence find the value of W. **1**

(ii) Hence, verify that W is also given by $W = (\underset{\sim}{F} \cdot \hat{\underset{\sim}{s}})\,|\underset{\sim}{s}|$. **1**

(iii) Find the **vector projection of** $\underset{\sim}{F}$ **onto** $\underset{\sim}{s}$. **1**

(d) The diagram shows the graph of $y = \dfrac{1}{x^2+1}$ and the graph of $y = 1 - \dfrac{x}{2}$ for $0 \le x \le 1$.



(i) Find the exact volume of the solid of revolution formed when the region bounded by the graph of $y = \dfrac{1}{x^2+1}$, the y-axis and the line $y = \dfrac{1}{2}$ is rotated $360°$ about the y-axis. **2**

(ii) Find the exact volume of the solid of revolution formed when the region bounded by the graph of $y = 1 - \dfrac{x}{2}$, the y-axis and the line $y = \dfrac{1}{2}$ is rotated $360°$ about the y-axis. **2**

(iii) Use the results from parts (c) (i) and (ii) to show that $\ln 2 > \dfrac{2}{3}$. **1**

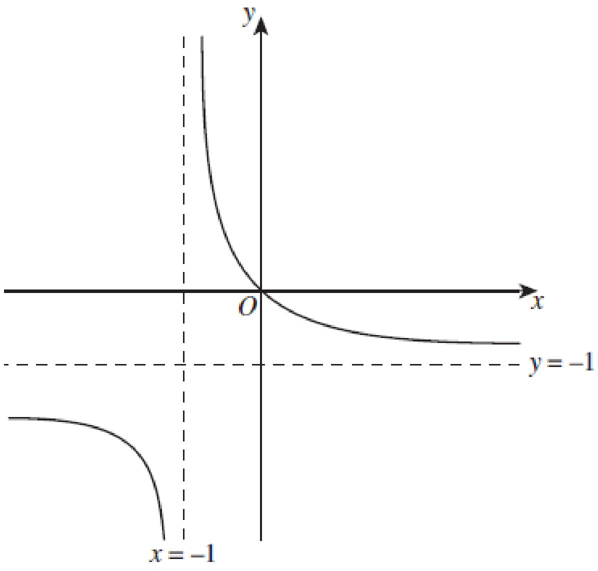
End of Question 12

Question 13 (15 marks) Start a new writing booklet

(a) Prove by mathematical induction that, for all integers $n \geq 1$, **3**

$$\frac{2}{1\times 3}+\frac{2}{2\times 4}+\frac{2}{3\times 5}+...+\frac{2}{n(n+2)}=\frac{3}{2}-\frac{2n+3}{(n+1)(n+2)} .$$

(b) The diagram below is a sketch of the graph of the function $f(x)=\frac{-x}{x+1}$.



(i) Sketch the graph of $y=-f(x)$, showing all asymptotes and intercepts. **2**

(ii) Sketch the graph of $y=\left(f(x)\right)^2$, showing all asymptotes and intercepts. **2**

(iii) Solve the equation $\left(f(x)\right)^2 = f(x)$ for $\boldsymbol{x}$. **1**

Question 13 continues on page 11

Question 13

(c) The area $A\ cm^2$ is occupied by a bacterial colony. The colony has its growth modelled by

The logistic equation $\frac{dA}{dt} = \frac{1}{25}A(50 - A)$ where $t \geq 0$ and t is measured in days.

At time $t = 0$, the area occupied by the bacteria colony is $\frac{1}{2}\ cm^2$.

(i) Show that $\frac{1}{A(50-A)} = \frac{1}{50}\left(\frac{1}{A} + \frac{1}{50-A}\right)$. **1**

(ii) Using the result from part (c) (i), solve the logistic equation and **3**

hence show that $A = \frac{50}{1+99e^{-2t}}$.

(iii) According to this model, what is the limiting area of the bacteria colony? **1**

(iv) Find the exact time when the rate of change in the area occupied by the bacterial colony is **2**

at its maximum.

End of Question 13

Question 14 (15 marks) Start a new writing booklet

(a) At time t seconds the length of the side of a cube is x cm, the surface area of the cube is $S\ cm^2$, and the volume of the cube is $V\ cm^3$. The surface area of the cube is increasing at a constant rate of $8\ cm^2/s$.

(i) Show that $\frac{dx}{dt}=\frac{k}{x}$ where k is a constant. **2**

(ii) Show that $\frac{dV}{dt}=2V^{\frac{1}{3}}$. **2**

(iii) Given that $V=8$ when $t=0$, find the value of t when $V=16\sqrt{2}$. **2**

(b)

(i) Prove the trigonometric identity $\tan 3\theta=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$. **3**

(ii) Use the identity from part (b) (i) to show that the roots of the cubic equation $x^3-3x^2-3x+1=0$ are $\tan\frac{\pi}{12}, \tan\frac{5\pi}{12}$ and -1. **2**

(iii) Hence, show that $\tan\frac{\pi}{12}=2-\sqrt{3}$. **2**

(c) The table shows selected values of a one-to-one differentiable function $g(x)$ and its derivative $g'(x)$. **2**

x	–1	0
$g(x)$	–5	–1
$g'(x)$	3	$\frac{1}{2}$

Let $f(x)$ be a function such that $f(x)=g^{-1}(x)$. Find the value of $f'(-1)$.

End of paper

BLANK PAGE

Hornsby Girls High School
Year 12 Mathematics Extension 1 HSC Trial 2020
Solutions

Solutions	Marker's Comments
Question 1 **A** If α and $\alpha + 1$ are zeros of $P(x)$, then $P(x) = x^2 - (2\alpha + 1)x + (\alpha^2 + \alpha)$. Equating coefficients gives $b = -(2\alpha + 1)$ and $c = \alpha^2 + \alpha$.	$\frac{52}{69}$ were correct.
Question 2 **D** $\frac{d}{dx}(\tan^{-1} f(x)) = \frac{f'(x)}{1 + (f(x))^2}$ $f(x) = 2x - 1$ and $f'(x) = 2$ $\frac{d}{dx}(\tan^{-1}(2x - 1)) = \frac{2}{1 + (2x - 1)^2}$ $= \frac{2}{4x^2 - 4x + 2}$ $= \frac{1}{2x^2 - 2x + 1}$	$\frac{54}{69}$ were correct.
Question 3 **C** Since there are 80 students, they could be divided into 10 groups each with 8 students. To ensure that at least one group has 9 students, we need 9 groups.	$\frac{46}{69}$ were correct.
Question 4 **A** $R\sin(x + \alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$ $\sin x + \cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$ Equating coefficients of $\sin x$ gives $R\cos\alpha = 1$. (1) Equating coefficients of $\cos x$ gives $R\sin\alpha = 1$. (2) Squaring both (1) and (2) and adding gives $R^2 = 2 \Rightarrow R = \sqrt{2}$ (> 0). Substituting into (1) and (2) gives $\cos\alpha = \frac{1}{\sqrt{2}}$ and $\sin\alpha = \frac{1}{\sqrt{2}}$. So $\alpha = \frac{\pi}{4}$ and hence $\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$.	$\frac{56}{69}$ were correct.

Solutions	Marker's Comments
Question 5 **A** The table outlines the possible seating arrangements. M1 | M2 | M3 | M4 | F1 | F2 | F3 | F4 1 | 3 | 2 | 1 | 4 | 3 | 2 | 1 Therefore the number of possible seating arrangements is $3! \times 4! = 144$.	$\frac{55}{69}$ were correct.
Question 6 **C** At $(0, 0)$, $\frac{dy}{dx} = 0$ and so **A** is incorrect. At $(-1, 1)$, $\frac{dy}{dx} = 0$ and so **B** and **D** are incorrect.	$\frac{42}{69}$ were correct. Many chose B wrongly.
Question 7 **B** $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$ $= \frac{1}{2}\overrightarrow{OB} + \overrightarrow{AB}$ $= \frac{1}{2}\overrightarrow{OB} + \overrightarrow{AO} + \overrightarrow{OB}$ $= \frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a} + \underset{\sim}{b}$ $= \frac{3}{2}\underset{\sim}{b} - \underset{\sim}{a}$	$\frac{57}{69}$ were correct.
Question 8 **C** $\int_{-1}^{1} \frac{k}{1+x^2}\, dx = 1$ $k(\tan^{-1} 1 - \tan^{-1}(-1)) = 1$ $k\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = 1$ $k = \frac{2}{\pi}$	$\frac{46}{69}$ were correct.

Solutions	Marker's Comments
Question 9 **C** The parametric equations are: $x = \cos^2 t \quad (1)$ $y = 4\sin^2 t \quad (2)$ $\frac{(2)}{4}$ gives $\frac{y}{4} = \sin^2 t. \quad (3)$ $(1) + (3)$ and using $\cos^2 t + \sin^2 t = 1$ gives $x + \frac{y}{4} = 1 \Rightarrow 4x + y = 4$. $0 \le \cos^2 t \le 1$ and so $0 \le x \le 1$. Therefore, $y = 4 - 4x$ for $0 \le x \le 1$.	$\frac{41}{69}$ were correct. Many chose B wrongly
Question 10 **C** The diagonals of $OABC$ are given by $\overrightarrow{OB}$ and $\overrightarrow{CA}$. To prove they are perpendicular, form $\overrightarrow{CA} \cdot \overrightarrow{OB}$ and show that it equals zero. $\overrightarrow{CA} = \underset{\sim}{a} - \underset{\sim}{c}$ and $\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{c}$. Therefore $\overrightarrow{CA} \cdot \overrightarrow{OB} = 0$ if $(\underset{\sim}{a} - \underset{\sim}{c}) \cdot (\underset{\sim}{a} + \underset{\sim}{c}) = 0$.	$\frac{58}{69}$ were correct.
Question 11 (a) (i) $f(x) = x^2 - 4x + 6$ is a parabola. Excluding the turning point at (2, 2), for each value of $f(x)$ in the range there are two x-values. Geometrically, this corresponds to a horizontal line intersecting the graph twice. If x and y are swapped, each x-value in the domain will have two y-values. Hence the inverse will not be a function.	1 mark Many students mistakenly thought that $f^{-1}(x)$ meant $(f(x))^{-1} = \frac{1}{f(x)}$ then gave a reason of restricting the domain to not include dividing by zero
(ii) Use the completing the square method to express $f(x)$ in turning point form: $f(x) = x^2 - 4x + 6 \quad (x \le 2)$ $= (x-2)^2 + 2$ Swap x and y, then make y the subject. $x = (y-2)^2 + 2$ $x - 2 = (y-2)^2$ $y - 2 = -\sqrt{x-2}$ ($\sqrt{x-2}$ is discarded as $y \le 2$) $y = -\sqrt{x-2} + 2$ $f^{-1}(x) = -\sqrt{x-2} + 2$	1 for interchanging x and y 1 for correct equation

Solutions	Marker's Comments
Question 11 (a) (iii) The domain is $x \ge 2$ as $x - 2 \ge 0$. The range is $y \le 2$ as $-\sqrt{x-2} \le 0$.	2 marks
(iv) The curves $y = f(x)$ and $y = f^{-1}(x)$ have a common intersection with the line $y = x$. For example, attempting to solve $f(x) = x$ for x: $x^2 - 4x + 6 = x$ $x^2 - 5x + 6 = 0$ $x = 2, 3$ When $x = 2$, $y = 2$ and so $(2, 2)$ lies on the line $y = x$. When $x = 3$, $y = 1$ and so $(3, 1)$ does not lie on the line $y = x$. Therefore the coordinates of P are $(2, 2)$.	2 marks
(b) Let $u = 1 + 2\tan x$. $\frac{du}{dx} = 2\sec^2 x = \frac{2}{\cos^2 x} \Rightarrow dx = \frac{\cos^2 x}{2} du$ When $x = 0$, $u = 1$ and when $x = \frac{\pi}{4}$, $u = 3$. $\int_0^{\frac{\pi}{4}} \frac{1}{(1 + 2\tan x)^2 \cos^2 x} dx = \int_1^3 \frac{1}{2u^2} du$ $= -\left[\frac{1}{2u}\right]_1^3$ $= -\left(\frac{1}{6} - \frac{1}{2}\right)$ $= \frac{1}{3}$	Generally well done. Some careless miskes.
(c) substitute $t = \tan\frac{x}{2}$, $\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} = -1$ $1 - t^2 - 2t = -1 - t^2$ $2t = 2$ $t = 1$ $\tan\frac{x}{2} = 1, \quad 0 \le x \le 2\pi$ $\frac{x}{2} = \frac{\pi}{4}, \quad 0 \le \frac{x}{2} \le \pi$ $x = \frac{\pi}{2}$ Also when $x = \pi$, $\cos x - \sin x = -1 - 0 = -1$ $\therefore x = \frac{\pi}{2}, \pi$	Most students used t-formula and find t correctly. Some students didn't check the domain and find the correct number of solutions in the domain. Some forgot to check the solution of π .

<table>
<tr><th>Solutions</th><th>Marker’s Comments</th></tr>
<tr><td>

Question 11

(d)

$u = 1 - x$

$\frac{du}{dx} = -1 \therefore dx = -du$

when $x = -3, u = 4$

when $x = 0, u = 1$

$\int_{-3}^{0} \frac{x}{\sqrt{1-x}} dx = \int_{1}^{4} \frac{1-u}{\sqrt{u}} (-du)$

$= \int_{1}^{4} u^{\frac{-1}{2}} - u^{\frac{1}{2}} du$

$= \left[2u^{\frac{1}{2}} - \frac{2}{3} u^{\frac{3}{2}} \right]_{1}^{4}$

$= \left(4 - \frac{16}{3}\right) - \left(2 - \frac{2}{3}\right)$

$= -\frac{4}{3} - \frac{4}{3}$

$= -\frac{8}{3}$

</td><td>

Generally well done.

Some careless mistakes.

</td></tr>
<tr><td>

Question 12

(a)

$F(x) = \int_{0}^{\frac{2}{3}} \frac{1}{\sqrt{4 - 9x^2}} \, dx$

$= \frac{1}{3} \left[\sin^{-1}\left(\frac{3}{2} x\right) \right]_{0}^{\frac{2}{3}}$

$= \frac{1}{3} \left[\sin^{-1} 1 - \sin^{-1} 0 \right]$

$= \frac{1}{3} \times \frac{\pi}{2}$

$= \frac{\pi}{6}$

Or

</td><td>

1 for recognising and manipulating to a form which can be integrated

1 for correct integration

1 for substitution and correct evaluation

</td></tr>
</table>

$F(x)=\int_0^{\frac{2}{3}}\frac{1}{\sqrt{4-9x^2}}\,dx$ $=\int_0^{\frac{2}{3}}\frac{1}{\sqrt{9(\frac{4}{9}-x^2)}}\,dx$ $=\int_0^{\frac{2}{3}}\frac{1}{3\sqrt{(\frac{4}{9}-x^2)}}\,dx$ $=\frac{1}{3}\left[\sin^{-1}\left(\frac{x}{\frac{2}{3}}\right)\right]_0^{\frac{2}{3}}$ $=\frac{1}{3}\left[\sin^{-1}\left(\frac{3}{2}x\right)\right]_0^{\frac{2}{3}}$ $=\frac{1}{3}\left[\sin^{-1}1-\sin^{-1}0\right]$ $=\frac{1}{3}\times\frac{\pi}{2}$ $=\frac{\pi}{6}$	
(b) (i) Using $\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$ with $A=B=\frac{\pi x}{8}$ gives: $\text{LHS}=\sin\frac{\pi x}{8}\sin\frac{\pi x}{8}$ $=\sin^2\frac{\pi x}{8}$ $\text{RHS}=\frac{1}{2}\left[\cos\left(\frac{\pi x}{8}-\frac{\pi x}{8}\right)-\cos\left(\frac{\pi x}{8}+\frac{\pi x}{8}\right)\right]$ $=\frac{1}{2}\left(\cos 0-\cos\frac{\pi x}{4}\right)$ $=\frac{1}{2}\left(1-\cos\frac{\pi x}{4}\right)$ So $\sin^2\left(\frac{\pi x}{8}\right)=\frac{1}{2}\left(1-\cos\frac{\pi x}{4}\right)$.	1 for correct rearrangement 1 for evaluation and simplifying Students need to show all working in show that questions eg cos0 should be included in the working here
Solutions	**Marker's Comments**
Question 12 (b) (ii) $A=6\int_0^8\sin^2\left(\frac{\pi x}{8}\right)dx$ $=3\int_0^8 1-\cos\frac{\pi x}{4}\,dx$ $=3\left[x-\frac{4}{\pi}\sin\frac{\pi x}{4}\right]_0^8$ $=3\left(8-\frac{4}{\pi}\sin 2\pi-(0-\sin 0)\right)$ $=3(8-0)$ $=24\quad unit^2$	Many students missed the 6 1 for rearrangement and integration 1 for substitution and correct evaluation

(c) (i) Substituting $\underset{\sim}{F} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $\underset{\sim}{s} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ into $W = \underset{\sim}{F} \cdot \underset{\sim}{s}$ gives: $W = \underset{\sim}{F} \cdot \underset{\sim}{s}$ $= \begin{pmatrix} 4 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ $= 20$	Part c was done well by most students 1
(ii) A unit vector in the direction of $\overrightarrow{PQ}$ is $\hat{\underset{\sim}{s}} = \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$. Substituting $\underset{\sim}{F} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, $\hat{\underset{\sim}{s}} = \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $\lvert \underset{\sim}{s} \rvert = 5$ into $W = (\underset{\sim}{F} \cdot \hat{\underset{\sim}{s}})\lvert \underset{\sim}{s} \rvert$ gives: $W = \left(\begin{pmatrix} 4 \\ -2 \end{pmatrix} \cdot \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}\right)5$ $= 20$	1
(iii) The component of $\underset{\sim}{F}$ in the direction of l is given by $\left(\frac{\underset{\sim}{F} \cdot \underset{\sim}{s}}{\underset{\sim}{s} \cdot \underset{\sim}{s}}\right)\underset{\sim}{s}$. Substituting $\underset{\sim}{F} \cdot \underset{\sim}{s} = 20$, $\underset{\sim}{s} \cdot \underset{\sim}{s} = 25$ and $\underset{\sim}{s} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ into $\left(\frac{\underset{\sim}{F} \cdot \underset{\sim}{s}}{\underset{\sim}{s} \cdot \underset{\sim}{s}}\right)\underset{\sim}{s}$ gives: $\left(\frac{\underset{\sim}{F} \cdot \underset{\sim}{s}}{\underset{\sim}{s} \cdot \underset{\sim}{s}}\right)\underset{\sim}{s} = \frac{20}{25}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ $= \frac{4}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ $= \begin{pmatrix} 2.4 \\ -3.2 \end{pmatrix}$ Alternatively, the component of $\underset{\sim}{F}$ in the direction of l is $(\underset{\sim}{F} \cdot \hat{\underset{\sim}{s}})\hat{\underset{\sim}{s}}$.	1 Some students had a scalar answer instead of a vector
Solutions	**Marker's Comments**
Question 12 (d) (i) Rearranging $y = \frac{1}{x^2 + 1}$ to express x^2 in terms of y gives $x^2 = \frac{1}{y} - 1$. $V = \pi \int_{\frac{1}{2}}^{1} \left(\frac{1}{y} - 1\right) dy$ $= \pi \Big[\ln\lvert y \rvert - y \Big]_{\frac{1}{2}}^{1}$ $= \pi\left(\ln 1 - 1 - \left(\ln\frac{1}{2} - \frac{1}{2}\right)\right)$ $= \pi\left(\ln 2 - \frac{1}{2}\right)$	When finding a volume about the y axis we need to use the form $V = \pi \int x^2 dy$ not $V = \pi \int y^2 dx$ Some students had trouble finding the correct expression for x^2 1 for correct integral statement 1 for correct integration and evaluation

(ii) Rearranging $y = 1 - \frac{x}{2}$ to express x in terms of y gives $x = 2(1 - y)$. $V = \pi\int_{\frac{1}{2}}^{1}(4(1-y)^2)dy$ $= -\frac{4\pi}{3}\left[(1-y)^3\right]_{\frac{1}{2}}^{1}$ $= -\frac{4\pi}{3}\left(0 - \frac{1}{8}\right)$ $= \frac{\pi}{6}$ Alternatively: The solid formed is a cone of radius 1 and height $\frac{1}{2}$. Substituting these values into $V = \frac{1}{3}\pi r^2 h$ gives: $V = \frac{1}{3} \times \pi \times 1^2 \times \frac{1}{2}$ $= \frac{\pi}{6}$	Similar problems as part i 1 for correct integral statement 1 for correct integration and evaluation No-one used this approach
The volume of the solid under the curve $y = \dfrac{1}{x^2+1}$ is greater than the volume of the solid under the curve $y = 1 - \dfrac{x}{2}$, $\therefore V(i) > V(ii)$ (iii) From the diagram, it can be reasoned that $\pi\left(\ln 2 - \frac{1}{2}\right) > \frac{\pi}{6}$. So $\ln 2 - \frac{1}{2} > \frac{1}{6} \Rightarrow \ln 2 > \frac{2}{3}$.	1

<table>
<tr><th>Solutions</th><th>Marker's Comments</th></tr>
<tr><td>

Question 13

(a)

Consider $n = 1$.

$$\text{LHS} = \frac{2}{1 \times 3} = \frac{2}{3} \text{ and}$$

$$\text{RHS} = \frac{3}{2} - \frac{2(1)+3}{(1+1)(1+2)} = \frac{4}{6} = \frac{2}{3} = \text{LHS}.$$

The statement is true when $n = 1$.

Suppose true for $n = k$.

$$\text{So } \frac{2}{1 \times 3} + \frac{2}{2 \times 4} + \frac{2}{3 \times 5} + \ldots + \frac{2}{k(k+2)} = \frac{3}{2} - \frac{2k+3}{(k+1)(k+2)}.$$

Show it is true for $n = k+1$; that is,

$$\frac{2}{1 \times 3} + \frac{2}{2 \times 4} + \frac{2}{3 \times 5} + \ldots + \frac{2}{k(k+2)} + \frac{2}{(k+1)(k+3)} = \frac{3}{2} - \frac{2(k+1)+3}{((k+1)+1)((k+1)+2)}$$

$$\begin{aligned}
\text{LHS} &= \frac{2}{1 \times 3} + \frac{2}{2 \times 4} + \frac{2}{3 \times 5} + \ldots + \frac{2}{k(k+2)} + \frac{2}{(k+1)(k+3)} \\
&= \frac{3}{2} - \frac{2k+3}{(k+1)(k+2)} + \frac{2}{(k+1)(k+3)} \\
&= \frac{3}{2} - \frac{(2k+3)(k+3) - 2(k+2)}{(k+1)(k+2)(k+3)} \\
&= \frac{3}{2} - \frac{2k^2 + 7k + 5}{(k+1)(k+2)(k+3)} \\
&= \frac{3}{2} - \frac{(2k+5)(k+1)}{(k+1)(k+2)(k+3)} \\
&= \frac{3}{2} - \frac{2k+5}{(k+2)(k+3)} \\
&= \frac{3}{2} - \frac{2(k+1)+3}{((k+1)+1)((k+1)+2)} \\
&= \text{RHS}
\end{aligned}$$

If true for $n = k$, then true for $n = k+1$.

Hence, by mathematical induction, true for $n \geq 1$.

</td><td>

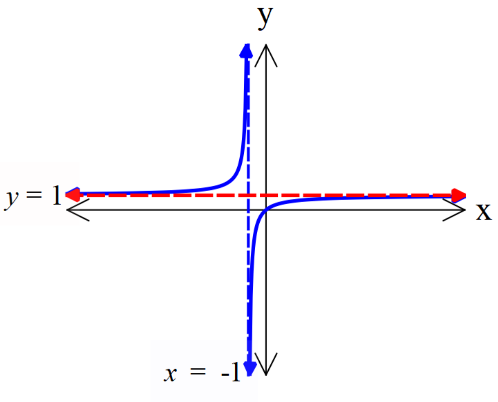
Most students followed the proving steps well and wrote the statement correctly.

Some made mistake in the algebra and end up "fudging" the result.

</td></tr>
<tr><td>

(b)

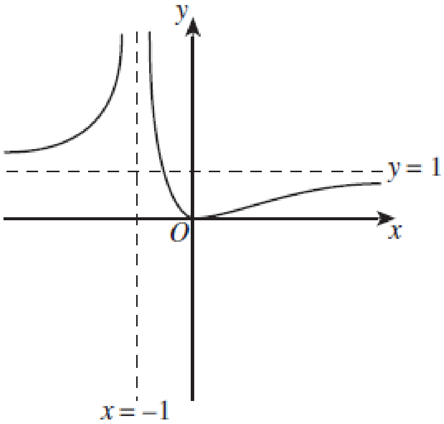
(i)



</td><td>

Most students were able to draw the correct asymptotes and shape of the curve.

</td></tr>
</table>

Solutions	Marker's Comments
Question 13 (b) (ii)  	Some students didn't recognise that this function is positive for all x. Some didn't find the correct asymptotes.
(iii) $(f(x))^2 = f(x) \Rightarrow f(x)(f(x)-1) = 0$ So $f(x) = 1$ or $f(x) = 0$. $-\frac{x}{x+1} = 1 \Rightarrow x = -\frac{1}{2}$ Hence $x = -\frac{1}{2}$ or $x = 0$. **OR** The graphs of $y = f(x)$ and $y = (f(x))^2$ intersect at O, where $x = 0$. The graphs of $y = f(x)$ and $y = (f(x))^2$ intersect on the line $y = 1$, where $x = -\frac{1}{2}$.	Some students missed the solution $x = 0$. Some students include the solution $x = -1$ which is not in the domain.
(c) (i) Start with the RHS and show that it equals the LHS. $\text{RHS} = \frac{1}{50}\left(\frac{(50-A)+A}{A(50-A)}\right)$ $= \frac{1}{A(50-A)}$ $= \text{LHS}$	Mostly well done.

Solutions	Marker's Comments
Question 13 (c) (ii) This is a differential equation of the form $\frac{dA}{dt} = g(A)$. Attempt to separate variables and integrate both sides. $\int 1\,dt = \int \frac{25}{A(50-A)}dA$ $t = \frac{1}{2}\int\left(\frac{1}{A} + \frac{1}{50-A}\right)dA$ (using the part (i) result) $= \frac{1}{2}(\ln\lvert A\rvert - \ln\lvert 50-A\rvert) + c$ $= \frac{1}{2}\ln\left\lvert\frac{A}{50-A}\right\rvert + c$ Rearranging gives $A_0e^{2t} = \frac{A}{50-A}$ where $A_0 = e^{-2c}$ and hence $A_0 > 0$. When $t = 0$, $A = \frac{1}{2}$ and so $A_0 = \frac{1}{99}$. *Note: There are various possible ways to find the value of the constant.* $e^{2t} = \frac{99A}{50-A}$ $99Ae^{-2t} = 50 - A$ $A(1 + 99e^{-2t}) = 50$ So $A = \frac{50}{1 + 99e^{-2t}}$.	Generally well done. Some mistakes solving the DE and finding the constant.
(iii) As $t \to \infty$, $1 + 99e^{-2t} \to 1$ and so $A \to \frac{50}{1} = 50$. The limiting area of the bacteria colony is 50 cm^2.	Generally well done.

Solutions	Marker's Comments
(c) (iv) The graph of $\frac{dA}{dt}$ versus A (inverted parabola) has a maximum at $A = 25$. It requires us to find the value of t such that $25 = \frac{50}{1+99e^{-2t}}$. $25(1+99e^{-2t}) = 50$ $1+99e^{-2t} = 2$ $e^{-2t} = \frac{1}{99}$ $e^{2t} = 99$ $t = \frac{1}{2}\ln 99$ (days) The rate of change of the area is at its maximum at $t = \frac{1}{2}\ln 99$ (days). *Note: There are other valid but more time-consuming methods of determining this solution.* *Method 1:* *Finding* $\frac{d^2A}{dt^2} = \frac{1}{25^2}A(50-A)(50-2A)$, *determining that* $\frac{dA}{dt}$ *is a maximum when* $A = 25$ *and then solving for t as above.* *Method 2:* *Determining the value of t when the (non-stationary) point of inflection occurs by finding* $\frac{d^2A}{dt^2}$ *in terms of t and then finding the value of t such that* $\frac{d^2A}{dt^2} = 0$.	Generally well done. Mark is not granted for carry-forward error trying to solve $e^{2t} = 0, \quad \therefore t = 0$

Solutions	Marker's Comments
Question 14 (a) (i) Surface area of a cube with a side length of x is $6x^2$ $S = 6x^2$ $\frac{dS}{dx} = 12x$ Given $\frac{dS}{dt} = 8$ $\frac{dx}{dt} = \frac{dS}{dt} \div \frac{dS}{dx} = \frac{8}{12x}$ $= \frac{\frac{2}{3}}{x}$ with $k = \frac{2}{3}$	Some careless mistake of $S = x^2$. "Show" that $\frac{dx}{dt} = \frac{k}{x}$: should express in that format.
(ii) $V = x^3$ (volume of a cube with side length x) $\frac{dV}{dx} = 3x^2$ $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$ $= 3x^2 \times \frac{2}{3x} = 2x$ $x = \frac{1}{2}\frac{dV}{dt}$ $x = V^{\frac{1}{3}}$ $\frac{1}{2}\frac{dV}{dt} = V^{\frac{1}{3}}$ $\frac{dV}{dt} = 2V^{\frac{1}{3}}$	Many had problem in obtaining the required expression.
(iii) $\frac{dV}{dt} = 2V^{\frac{1}{3}}$ $V^{-\frac{1}{3}}dV = 2\mathrm{dt}$ $\int V^{-\frac{1}{3}}\,dV = \int 2\,\mathrm{dt}$ $\frac{3}{2}V^{\frac{2}{3}} = 2t + C$ Given that $V = 8$ when $t = 0$ $\frac{3}{2}8^{\frac{2}{3}} = 2 \times 0 + C$ or $C = 6$ $\frac{3}{2}V^{\frac{2}{3}} = 2t + 6$ Find t when $V = 16\sqrt{2}$ $\frac{3}{2}(16\sqrt{2})^{\frac{2}{3}} = 2t + 6$ $12 = 2t + 6$ $t = 3$	Many had problem with evaluating $\left(16\sqrt{2}\right)^{\frac{2}{3}}$.

Solutions	Marker's Comments
Question 14 (b) (i) Use of $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$ with $A=B=\theta$. $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$ Use of $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$ with $A=2\theta$ and $B=\theta$. $\tan 3\theta=\dfrac{\tan 2\theta+\tan\theta}{1-\tan 2\theta\tan\theta}$ $=\dfrac{\dfrac{2\tan\theta}{1-\tan^2\theta}+\tan\theta}{1-\dfrac{2\tan\theta}{1-\tan^2\theta}\tan\theta}$ $=\dfrac{\dfrac{2\tan\theta+\tan\theta(1-\tan^2\theta)}{1-\tan^2\theta}}{\dfrac{(1-\tan^2\theta)-2\tan^2\theta}{1-\tan^2\theta}}$ $=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$	Generally done well. Students need to know the formula for $\tan 2\theta$. A few did not complete their working to the required form.
(ii) Consider $x^3-3x^2-3x+1=0$ with $x=\tan\theta$. $\tan^3\theta-3\tan^2\theta-3\tan\theta+1=0$ $\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}=1$ So $\tan 3\theta=1$ and finding the roots of $\tan 3\theta=1$ corresponds to finding the roots of the cubic equation where $x=\tan\theta$. $3\theta=\tan^{-1}1+k\pi$ where k is an integer $\theta=\dfrac{\pi}{12}+\dfrac{k\pi}{3}$ $=\dfrac{\pi}{12},\dfrac{5\pi}{12},\dfrac{9\pi}{12}$	Many did not relate the similarity in structure of $x^3-3x^2-3x+1=0$ to $\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$. Many did not connect that $x^3-3x^2-3x+1=0$ has three roots, neither did they realise that $\tan 3\theta$ indicates there are 3 roots to the solution.

Solutions	Marker's Comments
Question 14 (b) (iii) $\tan\frac{3\pi}{4} = -1$ and so one factor of the cube is $x+1$. So $x^3 - 3x^2 - 3x + 1 = (x+1)(x^2 - 4x + 1)$. So $\tan\frac{\pi}{12}$ and $\tan\frac{5\pi}{12}$ are the roots of $x^2 - 4x + 1 = 0$. Solving the quadratic equation $x^2 - 4x + 1 = 0$ for x gives $x = 2 \pm \sqrt{3}$. Since $\tan\frac{\pi}{12} < \tan\frac{5\pi}{12}$, $\tan\frac{\pi}{12}$ is the smaller root and $x = 2 - \sqrt{3}$.	Students have not stated succinctly for the rejection of $2+\sqrt{3}$ as a solution for $\tan\frac{\pi}{12}$.
(c) From the table, $f(x) = g^{-1}(x)$ and so $f(-1) = g^{-1}(-1) = 0$. $f'(-1) = \frac{1}{g'(f(-1))}$ $= \frac{1}{g'(0)}$ $= \frac{1}{\frac{1}{2}}$ $= 2$	This question is very poorly executed. It is a pity the lack of using graph as a tool to solve this.

End of Paper